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The Mutual Independence
of Hilbert's Axioms

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THE
MUTUAL INDEPENDENCE
OF
HILBERT'S AXIOMS
WITHIN
THE VARIOUS GROUPS

BY
ARTHUR TIMOTHY BELL, A. B., 1901.

Thesis for the Degree of Master of Arts
in the Graduate School.

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1902.

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May 29 1902

THIS IS TO CERTIFY THAT THE THESIS PREPARED UNDER MY SUPERVISION BY

Mr. Arthur Timothy Bell

ENTITLED Mutual Independence of Hilbert's Axioms.

IS APPROVED BY ME AS FULFILLING THIS PART OF THE REQUIREMENTS FOR THE DEGREE

OF Master of Arts

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HEAD OF DEPARTMENT OF Mathematics



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CHAPTER I

INTRODUCTION.

§1. Outline of Thesis.

Chapter I of my thesis gives a statement of that part of Hilbert's work which bears directly upon my investigations. My object in giving this is to show the foundations for my research work. A statement of my problem and method of procedure are here made and given concreteness by examples from Hilbert, whom I have followed in my manner of attack.

In Chapter II, I have presented the individual methods for showing the independence of each of Hilbert's axioms within the various groups.

Chapter III contains some of Hilbert's research into the importance of certain theorems and a brief account of his algebra of segments, without which the thesis is not a completed treatise. A short discussion of Schur's and Moore's papers, which bear on this line of mathematics, is also given here.

§2 General Statement of Problem and Bibliography.

Since my thesis is so dependent upon the work of Professor Hilbert in his "Grundlagen der Geometrie", it is necessary in order to make my thesis intelligible that I discuss at some length this treatise.

His lectures on this subject were first given during the winter semester 1898-99. They were afterwards revised and re-arranged and published as an address commemorating the unveiling of the Gauss-Weber monument at Göttingen June 17, 1900. Professor Hilbert treats of the foundations of geometry, or the underlying axioms which are necessary for the construction of our ordinary geometry. This task is by no means a new one, since from the time Euclid one of the greatest questions among geometers is what are the fundamental axiomatic principles indispensable for expression of geometric relations. In recent years this critical study has received special investigations by some of our best mathematicians. Pasch* (1882) undertook to treat a pure geometry in a strict sense as a purely deductive science based on certain assumed abstract notions which must conform to certain assumed relations. He uses as the basal elements for geometry the point, the linear segment, and the planar segment. In terms of these elements he defines the line and the plane. Certain sets of points are the component parts of the linear segment and the planar segment. Peano* (1889, 1894) also did considerable work on the same line of investigation. He retained as elements of geometry the point and the linear segment, but defined the plane in terms of these elements alone. Ingrami* (1889), following Peano, simplified his system further, but left the final perfecting of this master work to Hilbert. Since the appearance of Hilbert's treatise, Schur* and Moore* have written critical articles on Hilbert's work, in which they discuss certain redundancies of Hilbert's axioms as they see it. These papers will be discussed at some length in Chapter III of this thesis.

* See Bibliography at the end of this section.

The object of geometry we recognize as the analysis and description of our space intuition. These abstractions lead us to the conception of three systems of things; points, lines, and planes, which form the basis of our geometrical ideas. By the use of definitions these elements are given certain correlations for which geometry seeks to establish general laws. In order that we may build up an entirely consistent system of propositions certain requirements, known as axioms, must be satisfied in all conceivable relations between the elements. The selection of these axioms and investigation into their relation to each other form the basis of Hilbert's "Grundlagen der Geometrie". In his treatise a new attempt is made to choose for geometry a simple and complete set of independent axioms.

I give at this point a list of the leading literature on this subject.

PASCH, Vorlesungen über neuere Geometrie, Leipzig, 1882.

PEANO, I principii di Geometria, 1889 and Sui fondamenti della Geometria, Rivista di Matematica, v. 4 (1894), pp.51-90.

INGRAMI, Elementi di Geometria per le scuole secondarie superiori, Bologna, 1899.

SCHUR, Ueber die Grundlagen der Geometrie, Mathematische Annalen, v. 55 (1901), pp. 265-292.

HOLDER, Anschauung und Denken in der Geometrie, Leipzig, 1900.

SOMMER, Hilbert's foundations of Geometry, Bulletin of American Mathematical Society, vol.6 (1900), p.292.

MOORE, On the Projective Axioms of Geometry, Paper before American Mathematical Society, Dec. 28, 1901.

HILBERT, Grundlagen der Geometrie, given at unveiling of the Gauss-Weber monument, and his lectures on Euclidean Geometry, winter semester 1898-99. Professor Townsend of the University of Illinois is now preparing for publication an English translation of Hilbert's Grundlagen der Geometrie.

§3. Brief Statement of Hilbert's Work.

From our definition of an axiom as a self evident truth, it follows that the axioms, necessary for the construction of geometry, must have immediate general validity not capable of demonstration. They must also form a system of propositions independent of each other and not further reducible. They must, furthermore, be in no way contradictory one with the others. Such axioms serve as a basis to make any geometrical definition possible; in other words, a thinkable definition gains its meaning only when it can be shown by the aid of axioms that the definition has a real content. Another thing which we include in the idea of axiom is that it shall be simple; that is, that the least number of axioms be employed to establish and circumscribe the relations existing between the elements. Hence ~~none~~ of the axioms should have redundancies in them, or we may say, none should be a ~~co~~llary following from the remaining axioms. Still another requirement of a system of axioms is that of completeness such that all relations of our spatial intuition are satisfied. It is possible to detach (as Professor Hilbert has shown in his researches) certain axioms from a complete system and yet have a system sufficient for forming a logical and consistent system of

geometry in which no contradictions arise.

Professor Hilbert has built up a system of axioms which he claims is a simple and complete set of independent axioms. This statement is accepted by Hölder and Sommer, but it is questioned by Schur and Moore, who claim that certain redundancies occur in Hilbert's system of axioms. I shall discuss the papers of Schur and Moore to some extent in my Chapter III. It is the purpose of Hilbert to make a basis for euclidean geometry, but beyond this to make his axioms a proper foundation for analytical geometry. His discussion on the applications to analytical geometry and on the use of important theorems, such as Pascal's and Desargues's theorems, will form a portion of Chapter III of this thesis.

Hilbert considers the elements of geometry as three systems of things, to be defined as we choose. The things of the first system he calls points and designates them by the letters A, B, C, - - -. The elements of the second system he calls straight lines and designates them by the letters a, b, c, - - -. The things of the third system are planes and designated by the Greek letters α , β , γ , - - -. Points are the elements of linear geometry; points and lines together are the elements of plane geometry; the elements of space consist of points, straight lines, and planes.

Our ideas of the relation between these elements we express by the use of the words "determine", "between", "parallel", "congruent", and "continuous". This exact description follows as a consequence of the axioms of geometry. The axioms themselves define the relations existing between our elements. Hilbert divides his system of axioms into five groups, each of which expresses, by itself, certain

fundamental facts of our intuition concerning the correlation of the geometrical elements.

The axioms are classified as follows:-

I,1-7 Axioms of connection.

II,1-5 Axioms of order.

III Axioms of parallels (Euclid's axiom).

IV,1-6 Axioms of congruence.

V. Axioms of continuity (Archimedes's axiom).

#4. List of Hilbert's Axioms.

In this section I shall give Hilbert's axioms as he arranged them in groups.

Group I. Axioms of connection.

This group of axioms has to do with the associations between our concepts, points, straight lines, and planes. These axioms are as follows:-

I,1. Two distinct points A and B always completely determine a straight line a. We write $AB=\underline{a}$, or $BA=\underline{a}$.

I,2. Any two distinct points of a straight line completely determine that line; that is, if $AB=\underline{a}$ and $AC=\underline{a}$, where $B \neq C$, then also $BC=\underline{a}$.

I,3. Three non-collinear points A, B, C always completely determine a plane α . We write $ABC=\alpha$.

I,4. Any three non-collinear points A, B, C of the plane α completely determine that plane.

I,5. If two points A, B of a straight line a lie in a plane α , then every point of a lies in α .

I,6. If two planes α, β have a point A in common, then they have at least a second point B in common.

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I,7. Upon every straight line there must exist at least two points, in every plane at least three non-collinear points, and in every space there must exist at least four non-coplanar points.

Axioms I,1,2,7, contain only statements concerning points and straight lines which are the elements of plane geometry. Hence these axioms are called the plane axioms of group I. The remaining axioms of this group are called space axioms since they deal with the elements of space (point, straight line, and plane).

Group II- Axioms of order.

This axiom group defines the relation commonly known as order of elements and gives meaning to our notion of "between". We state these axioms in the following manner:-

II,1. If A,B,C are points of a straight line and B lies between A and C, then B also lies between C and A.

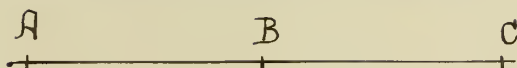


Fig. 1.

II,2. If A and C are two points of a straight line, then there exists at least one point B lying between A and C and at least one point D so situated that C lies between A and D.

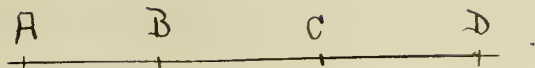


Fig. 2.

II,3. Of any three points situated on a straight line, there is one and only one which lies between the other two.

II,4. Any four points A,B,C,D of a straight line can always be

1. The first thing I noticed when I stepped out of the plane was the fresh air. It felt like a breath of life after being cooped up in a small space for hours. The sun was shining brightly, and the birds were chirping happily. I took a deep breath and smiled. This was my chance to see the world from a different perspective. I had heard so much about the beauty of the mountains and the kindness of the people. Now I was here, and it was all true. I felt like I had found a new home. I was going to stay here for a while, and I was going to make the most of it. I was going to explore every corner of this beautiful land and meet every person I could. I was going to live my life to the fullest. I was going to be happy.

2. The second thing I noticed was the food. It was so good, so different from what I was used to. I had heard that the food was amazing, and now I knew it was true. I had never tasted anything so delicious before. The chef was a local person, and he had a lot of experience. He had learned from his father, who had learned from his grandfather. The food was passed down through generations, and it was still as good as ever. I was going to eat every dish he had. I was going to taste everything. I was going to be happy.

3. The third thing I noticed was the people. They were so kind, so helpful. I had heard that the people were friendly, and now I knew it was true. They were always smiling, and they were always willing to help. I was going to be friends with everyone I met. I was going to be happy.

4. The fourth thing I noticed was the scenery. It was so beautiful, so peaceful. I had heard that the scenery was amazing, and now I knew it was true. The mountains were so high, and the water was so clear. I was going to take a lot of pictures. I was going to be happy.

so arranged that B shall lie between A and C and also between A and D, and, furthermore, so that C shall lie between A and D and also between B and D.

Definition. Hilbert defines a segment of a line as a system of two points A and B, lying upon the straight line in question. He denotes the segment by AB or BA. Points between A and B are points of AB.

II,5. Let A,B,C be three non-collinear points and let a be a straight line lying in the plane ABC and not passing through any of the points A,B,C. Then, if the straight line a passes through a point of the segment AB, it will also pass through either a point of the segment BC or a point of the segment AC.

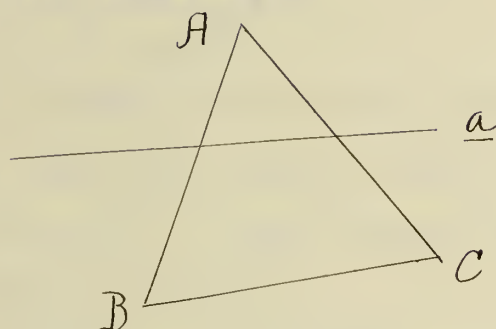


Fig. 3.

Axioms II,1-4 contain only statements concerning the points of a straight line, and hence we call them the linear axioms of group II. Axiom II,5 deals with the elements of the plane, and therefore is the plane axiom of the group.

These axioms are sufficient to prove that a straight line must contain an enumerable number of points. Hilbert gives as theorems, "Any straight line divides a plane into two regions," and "Any one point of a straight line divides the line into two half-rays." With this material Hilbert defines a polygon as a broken line consisting of a

and a number of other things which I have not time to mention here. I have also written a number of papers on the subject of the history of the United States, and I have also written a number of papers on the subject of the history of the United States.

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 The history of the United States is a subject of great interest to all who are interested in the history of the world. It is a subject which has attracted the attention of many of the greatest writers of the world, and it is a subject which has been the subject of many of the most important works of the world.

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system of segments AB, BC, CD, \dots, KL , where L and A coincide. For brevity he names the polygon by $ABC\dots KA$. The polygon is said to be a simple polygon if the points A, B, C, D, \dots, K are distinct, and if no two of the sides AB, BC, CD, \dots, KA , have a point in common aside from the vertices between two adjacent segments. The vertices are the points A, B, C, \dots, L . Hilbert proves as a property of a simple polygon that it divides the plane into two regions. If the polygon have only three sides, it is called a triangle. In Fig. 3, ABC is a triangle, and axiom II, 5 is frequently referred to as the triangle-transversal axiom.

Group III. Axiom of parallels (Euclid's Axiom).

By bringing this axiom into geometry, we make the fundamental principles of geometry much simpler and their development much easier. This axiom is expressed as follows:-

III. In a plane α there can be drawn through any point A , lying outside of a straight line a , one and only one straight line which does not intersect the line a . This straight line is called the parallel to a through the given point A .

The latter assertion (only one) is essential, since Hilbert shows that the first statement may be deduced from the other axioms.

Group IV. Axioms of congruence.

By means of the axioms of this group we circumscribe our idea of congruence existing between segments or angles.

IV, 1. If A, B are two points on a straight line a and if A' is a point upon the same or another straight line a' , then upon a given side of a straight line a' from A' , we can find one and only one point B' such that the segment AB or BA is congruent to the segment $A'B'$. We indicate this relation by writing

$$AB \equiv A'B'.$$

Every segment is congruent to itself; that is, we always have

$$AB \equiv AB.$$

IV,2. If a segment AB is congruent to a segment $A'B'$ and also to the segment $A''B''$, then the segment $A'B'$ is congruent to the segment $A''B''$; that is, if $AB \equiv A'B'$ and $AB \equiv A''B''$, then $A'B' \equiv A''B''$.

IV,3. Let AB and BC be two segments of a straight line \underline{a} which have no points in common aside from the point B , and, furthermore, let $A'B'$ and $B'C'$ be two segments of the same or of another straight line $\underline{a'}$, having likewise no point other than B' in common. Then, if $AB \equiv A'B'$ and $BC \equiv B'C'$ we have $AC \equiv A'C'$.

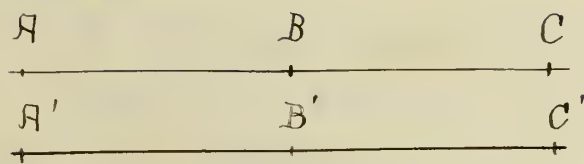


Fig. 4.

Definition. Hilbert defines an angle much in this manner. Let α be any arbitrary plane and h, k any two distinct half-rays lying in α and emanating from the point O so as to form each a part of two different lines. This system formed by the two half-rays h, k Hilbert calls an angle and represents it by the symbol $\angle(h, k)$ or $\angle(k, h)$.

IV,4. Let an angle (h, k) be given in the plane α and let a straight line \underline{a} be given in a plane α' . Suppose also that, in the plane α' , a definite side of the straight line $\underline{a'}$ be assigned. Denote by h' a half-ray of the straight line $\underline{a'}$ emanating from a point O' of this line. Then in the plane α' there is one and only one half-ray k' such that the angle (h, k) or (k, h) is congruent to the angle (h', k')

and at the same time all interior points of the angle (h',k') lie upon the given side of $\underline{a'}$. We express this relation by means of the notation

$$\angle(h,k) \equiv \angle(h',k').$$

Every angle is congruent to itself; that is,

$$\angle(h,k) \equiv \angle(h,k),$$

$$\text{or } \angle(h,k) \equiv \angle(k,h).$$

IV,5. If the angle (h,k) is congruent to the angle (h',k') and to the angle (h'',k'') , then the angle (h',k') is congruent to the angle (h'',k'') ; that is to say, if $\angle(h,k) \equiv \angle(h',k')$ and $\angle(h,k) \equiv \angle(h'',k'')$, then $\angle(h',k') \equiv \angle(h'',k'')$.

IV,6. If in the two triangles ABC and $A'B'C'$ the congruences $AB \equiv A'B'$, $AC \equiv A'C'$, $\angle BAC \equiv \angle B'A'C'$ hold, then the congruences,

$$\angle ABC \equiv \angle A'B'C' \text{ and } \angle ACB \equiv \angle A'C'B' \text{ also hold.}$$

The linear axioms of this group are IV,1-3, the plane axioms are IV,4-6.

Group V. Axiom of continuity (Archimedes's Axiom).

This axiom introduces into geometry the notion of continuity. We base this idea upon a convention relating to the equality of two segments. Two segments are defined as equal if they be congruent in the sense of the axioms of group IV. Using this convention, Hilbert states the axiom of Archimedes's in this manner:-

V. Let A_1 be any point upon a straight line between the arbitrarily chosen points A and B . Take the points A_2, A_3, A_4, \dots so that A_1 lies between A and A_2 , A_2 between A_1 and A_3 , A_3 between A_2 and A_4 , etc.

Moreover, let the segments

$$AA_1, A_1A_2, A_2A_3, A_3A_4, \dots$$

be equal to each other. Then among this series of points there always exists a certain point A_n such that B lies between A and A_n .

This axiom of Archimedes is a linear axiom, since it deals with the continuity of segments of lines.

We have now given Hilbert's complete system of axioms which contains in all eight linear, seven plane, and five space axioms.

5. Compatibility and Mutual Independence of Hilbert's Axioms.

As previously stated Hilbert maintains that the axioms just given are not contradictory to one another, since it is not possible to deduce from these axioms by any logical process of reasoning a proposition which contradicts certain others of the system.

To demonstrate this assertion, Hilbert builds up a geometry where all of the axioms hold. This is based upon a number system of a certain domain \mathcal{N} , consisting of all algebraic numbers which may be obtained by beginning with the number one and applying to it a finite number of times the operations of addition, subtraction, multiplication, division, and the irrational operation $\sqrt{1+\omega^2}$, where ω represents a number arising from the five operations already given.

From this system he constructs a geometry by letting a pair of numbers (x, y) define a point, and the ratio of three such numbers $(u:v:w)$, where w, v are not both equal to zero, define a straight line. He lets the

$$ux+vy+w=0$$

express the condition that the point (x, y) lies on the straight line

(u:v:w). Then by making proper conventions, he shows that all the five groups remain valid in this geometry, and hence the axioms are non-~~con~~contradictory as we have a system in which all the axioms are true.

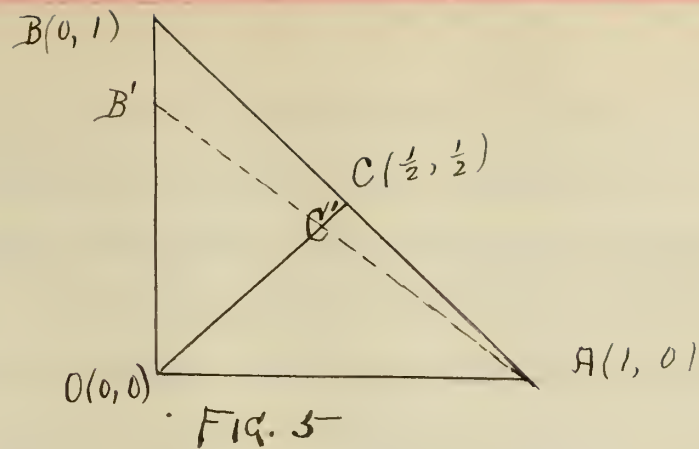
Hilbert, furthermore, claims that his axioms are mutually independent*, for he maintains that it may be shown that none of the axioms can be deduced from the remaining ones by any logical reasoning. If we can build up a system of geometry in which certain axioms hold while certain other axioms do not hold, we have at once that the axioms not holding are not dependent upon the set of axioms which do hold. This is the method used by Hilbert. He proves the independence of a group of his axioms by building up a geometry in which four groups remain valid while the remaining group of axioms does not hold. This establishes the independence of the one group from the other four groups.

Hilbert shows, for example, the independence of group III by building up a system of geometry (non-euclidean) in which all of his axioms hold except axiom III. The points of this geometry are restricted to a sphere, and hence any number of straight lines can be drawn through a point in the same plane of a given line without cutting the given line. The other axioms of Hilbert's system hold, while axiom III fails. Hence axiom III is independent of axioms I, 1-7, II, 1-5, IV, 1-6, and V.

To show the independence of the axioms of congruence (group IV), Hilbert constructs the following geometry. In this system he defines the length of a segment to be $\sqrt{(x_1 - x_2 + y_1 - y_2)^2 + (y_1 - y_2)^2 + (z_1 - z_2)^2}$, and segments having equal lengths according to this convention shall be congruent.

ment

*The results of Schur and Moore are at variance with this statement. Their work is considered in my chapter III.



In the figure given above, the triangles OBC and OAC have their angles congruence in the ordinary sense, but $OA=1$ and $OB=2$ and are not congruent in this geometry. This violates Hilbert's first theorem of congruence. If we should choose B' so that $OB' \equiv OA$, then we see at once that the angle $OB'C'$ and angle OAC' are not congruent and also $OC'B \equiv OC'A$. Hence axiom IV,6 is not valid for we have two triangles $OB'C'$ and OAC' which have $\angle AOC' \equiv \angle B'OC'$, and $OB' \equiv OA$ and $OC' \equiv OC'$, while the congruences $\angle OC'B \equiv \angle OC'A$ and $\angle OAC' \equiv \angle OB'C'$ do not hold. The other groups of axioms are valid, and hence Hilbert has by this system shown that axiom IV,6 is not a consequence of the other groups of axioms. By making other conventions as to the meaning of congruence, the independence of the other axioms of group IV may be shown. This work I have done, and it will be given in #4 of chapter II of this thesis.

To show the independence of the axiom of Archimedes we must produce a geometry in which all the axioms are fulfilled with the exception of the one in question. Hilbert does this by constructing a number system consisting of a certain domain $\mathcal{A}(t)$ of all those algebraic functions of t which may be obtained from t by means of the four arithmetical operations of addition, subtraction, multiplication, division, and the fifth operation $\sqrt{1+\omega^2}$, where ω represents any function arising from applying these five operations a limited number of

times. Making conventions corresponding to those of domain \mathcal{B} for the numbers of the system $\mathcal{B}(t)$ to have them related so as to conform to our notion of points, lines, and planes and their correlations, we get a system of geometry in which axioms of the first four groups hold. Axiom V, however, does not hold; for, if \underline{n} is any arbitrary positive rational integral number, then, for two numbers \underline{n} and \underline{t} of this number system $\underline{n} < \underline{t}$ certainly holds. This is true, because the difference $\underline{n} - \underline{t}$, when considered as a function of t is always negative for sufficiently large values of \underline{t} . This fact can be expressed in the following manner:- The two numbers \underline{n} and \underline{t} of the domain $\mathcal{B}(t)$, each being greater than zero, have the property that any multiple whatever of the first number must always remain less than the second number. This number system therefore gives us a non-archimedean geometry, for a given part of a segment of a line, expressible in this number system, cannot be made equal to the entire segment by any number of multiplications. Therefore axiom V is independent of the other four groups, as we have a system of geometry in which the first four axiom groups hold while axiom V does not remain valid.

These cases serve to show Hilbert's method of demonstrating the independence of a group of axioms, which method is nothing more or less than constructing a system of geometry in which four groups hold while the remaining group fails, thus establishing its independence.

#6. Hilbert's Work in Establishing the Independence Within the Various Groups.

In his lecture on "Euclidean Geometry" during the winter semester 1898-'99, Hilbert went further into the analysis of independence

of his axioms, saying that there existed a mutual independence of the axioms within each group. He gave the systems of geometry which follow in this section to prove the independence of axioms I,2; I,5; I,6.

It is the purpose of this thesis to build up the necessary systems of geometry in each case for proving the each axiom is not a consequence of the other axioms of the same group.

The mutual independence of any axiom may be established in the following manner. In order to show in general that any axiom A is not a consequence of the axioms B,C,D,---, it is sufficient to produce a geometry where B,C,D,---are valid, but where the axiom A does not hold. To illustrate the method Hilbert gives several geometries where all axioms of a group remain valid save one whose independence is thereby established. They are as follows:-

a) I,2 is not a consequence of I,1.

PROOF. In order to prove this, we will take as points in this new geometry all the integral, rational, positive numbers, and as straight lines the integral, rational, negative numbers. Hence no point is identical with a straight line. Let two points $A=p_1$ and $B=p_2$ determine a straight line according to the law

$$\left[\frac{p_1 p_2}{2} \right] = -g,$$

where $\left[\frac{p_1 p_2}{2} \right]$ indicates the greatest integral number less than the value of $\frac{p_1 p_2}{2}$. Then I,1 is fulfilled; for, having two points p_1, p_2 , the straight line g is uniquely determined. On the other hand I,2 is not fulfilled; for, if $A=1, B=2, C=3$, then $AB=-1, AC=-1$ and A, B, C lie on the same straight line. However, $BC=-3$, and hence B and C determine a line different from the line in which they lie, and hence I,2 does not hold. Therefore I,2 is independent of I,1.

Instead of the above convention as to points and straight lines, we might take as points all rational integral numbers, and as straight lines the pure imaginary numbers. Let the law of association of points and lines be defined by the relation

$$ip_1^2 p_2^2 = g.$$

Here again I,1 holds, but I,2 is not fulfilled. To illustrate Hilbert's proof further I may take the points $A=1, B=6$. They determine the line $36i$, or $AB=36i$. Similarly $C=2, D=3$ determine the same line, $CD=36i$. Hence A, B, C, D lie on the line $36i$. But $AC=4i$ and $BD=324i$, and hence I,2 breaks down in this system. But I,1 holds since two points uniquely determine a line.

Hence the truth of our proposition follows.

b) The axiom I,5 is independent of the other six axioms of group I.

PROOF. Take as points all points of euclidean space with exception of the single point 0. As straight lines of our new geometry, take the circles passing through the point 0. Planes shall be the ordinary euclidean planes. Then all the axioms of group I are fulfilled with the exception of I,5.

This system of Hilbert's, however, does not prove the independence of I,5 from all other axioms of group I, for I,3 and I,4 also break down in this system. The reason for this is at once obvious from the fact that three non-collinear points of this system, our lines being circles, may lie on an ordinary line, and hence leads to indeterminateness when the three determining points A, B, C happen to lie on the same ordinary straight line. In section 2 of chapter II, I shall give a proof for the independence of I,5, which meets all requirements of the case.

c) The axiom I,6 is independent of the remaining six axioms of the same group.

PROOF. We take, as points, the points of the euclidean space, excluding, however, all points of a certain straight line a except the point P. P shall belong to our system of points. As straight lines, take all the straight lines of space with the exception of the line a. Let the planes of our new geometry be the ordinary planes. Then all the axioms of the first group are fulfilled with the single exception of I,6. This is not fulfilled, because two planes in space, having the line a in common, have in this system of geometry thus defined only the point P in common. From this proof, it follows that I,6 is independent of the other axioms of the same group, since other axioms hold.

Hilbert's proof for the independence of II,4 will be given in the next chapter in # 3.

As stated at the beginning of this section my thesis has as its object the completing of this work of showing that each axiom is independent of the other axioms within the same group.

Having now a view of Hilbert's axioms and of his method of establishing the independence of axioms, we are now ready to proceed into my own investigations as given in chapter II.

CHAPTER II.

THE INDEPENDENCE of the AXIOMS within the GROUPS.

#1. Introduction to the Chapter.

In this chapter I give my original investigations of the mutual independence of Hilbert's axioms within the various groups.

of the same kind is mentioned in the following passage:

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As to method, I have held somewhat closely to Hilbert's manner of proof and have used several of his suggestions in building up my systems of geometry. In #6 of chapter I, I gave concrete examples of Hilbert's method of showing the independence of axioms within a group.

It is needless to state that in each system of geometry, used in demonstration, we have a geometry which is not of the ordinary kind, since it must fulfill the extraordinary condition of having at least one of the truths of our common geometrical conceptions invalid. In each new system of geometry, we must define anew the elements of the geometry, viz:— points, straight lines, and in the case of a space geometry, the planes also. The terms, "determine", "between", and "congruent" are likewise to be defined for the group of axioms which is concerned with the word in question, unless they be used in the ordinary sense.

Following Hilbert's method, I must, in proving the independence of a linear axiom of a group, build up a system in which only the other linear axioms of the group need hold. For the independence of a plane axiom, the system must fulfill other plane axioms of the group and also the linear axioms. In the case of a space axiom, all other axioms of the group must hold save the one whose independence we are investigating.

Since axiom groups III and V each consist of only one axiom, my work has to do only with groups I, II, and IV.

#2. Independence of Axioms within Group I.

We have as plane axioms of group I, axioms I, 1, 2, 7, which deal with points and lines, the elements of plane geometry. The remaining axioms of the group are space axioms, since the elements to which they relate are points, lines, and planes.

a) Axiom I,1 is not a consequence of axioms I,2,7.

PROOF. To establish this proposition, we build up the following new system of geometry. The points of the system shall be the points of the euclidean plane; our straight lines shall be circles passing through the origin, the origin, however, not being a point of any straight line of the system.

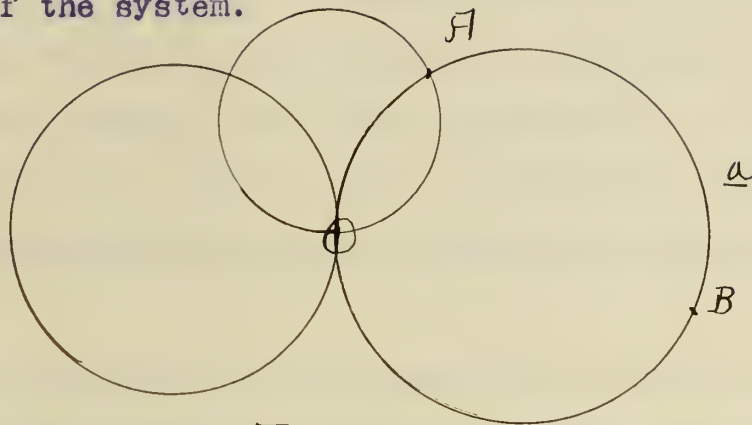


Fig. 6.

Then axiom I,1 does not hold, for we may choose as one point the origin, 0. In the figure here given we see that the points 0 and A do not determine the circle a which is a straight line of our system. Hence the general axiom that "any two points determine a straight line" breaks down in this system. We see, however, that I,2 holds, for two points of a line determine that line. Of the line a in Fig. 6, let us take the points A and B. The line a passes through 0 by definition. So we have as the line determined by A and B the line a, for it can be the only line which passes through A, B, and 0. The remaining plane axiom of the group, namely I,7, holds; for on any circle, which is a straight line of our system, there are an infinity of points. I need not show that space axioms of group I hold in this system, since we are concerned only with the plane axioms of the group. As before stated, in order to show the independence of a plane axiom, it is suf-

THE HISTORY OF THE UNITED STATES

THE HISTORY OF THE UNITED STATES, FROM THE FIRST SETTLEMENTS TO THE PRESENT TIME. BY JAMES M. SMITH, LL.D. VOL. I. NEW YORK: PUBLISHED BY J. B. LIPPINCOTT, 150 NASSAU ST. 1854.

THE HISTORY OF THE UNITED STATES, FROM THE FIRST SETTLEMENTS TO THE PRESENT TIME. BY JAMES M. SMITH, LL.D. VOL. I. NEW YORK: PUBLISHED BY J. B. LIPPINCOTT, 150 NASSAU ST. 1854.

ficient if I can build up a system of geometry where linear and other plane axioms of the group hold. Therefore the truth of our proposition is established as we have complied with these conditions.

The following statement, however, should be made here. An axiom is not fulfilled if it breaks down in a single case. In the system of geometry given above, any two points, chosen at random, determine a definite circle (straight line the system), providing that neither of the points is the origin. But if the origin be one of the chosen points, the axiom breaks down, and this is sufficient for our purpose.

b) The independence of axiom I, 2 from the other plane axioms of group I.

PROOF. We take as points of our geometry all points of the euclidean plane; i.e., ordinary points. Our straight lines shall be circles which shall be determined by two points in the following manner:- Let the circle, which by definition is the straight line of our geometry, be determined by the two points lying at the extremity of its diameter. Then I, 2 is not valid, for two points of a line will in general determine a line different from the line on which the points are taken. As shown in the figure here given,

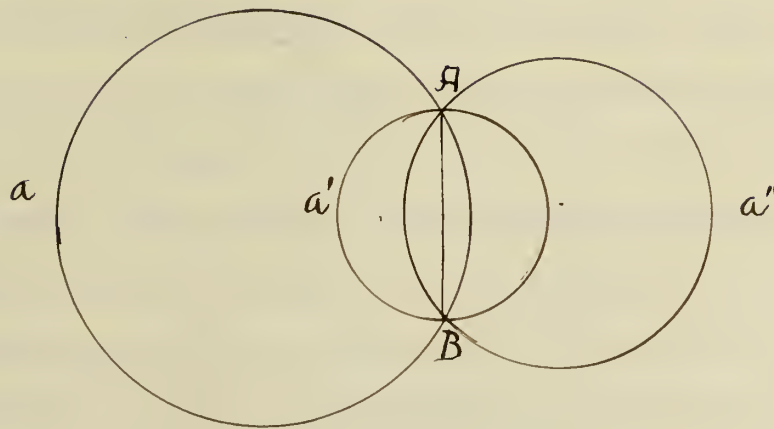


Fig. 7.

points A, B of the line a determine another line a'. On the other hand

I,1 holds, for in every case two points determine one and only one straight line of our system. The truth of I,7, is at once evident since every circle has an infinity of points lying on it. Therefore the independence of I,2 results from this system of geometry. As stated in paragraph a) of this section, we need not concern ourselves with the space axioms of the group, since we are dealing only with a plane geometry.

c) The independence of axiom I,3.

PROOF. Let us consider a geometry in which the points shall be all ordinary points of space; our straight lines shall be ordinary straight lines; and planes are those ordinary planes which pass through the origin. Then I,3 is not valid; for three non-collinear points, lying in an ordinary plane not passing through the origin, can not determine a plane of this system. But I,1,2,7, are valid, since our straight lines are of the ordinary kind. Hence any two points determine a line; two points of a line determine that line; and on every line there exists at least two points. As our planes are ordinary planes, three non-collinear points of a plane determine that plane, and axiom I,4 holds. Axiom I,5 is valid; for, if two points of a line lie in a plane, all points of the line lie in the plane, as lines and planes are of the common kind. I,6 is also true since two planes, having one point in common, will have an entire line in common. As our planes are the ordinary planes, they each have at least three non-collinear points. The space of this system, conforming to our usual space intuition, has at least four non-coplanar points. Hence I,7_{2,3} remain perfectly valid in this system. Therefore we have established that axiom I,3 is not a consequence of the other axioms of group I; since in this system of geometry just given I,3 breaks down, while the remaining axioms of the group are fulfilled.

d) Axiom I,4 is independent of the other axioms of group I.

PROOF. In this system of geometry, the points shall be the ordinary points of space, and our lines shall be ordinary straight lines. The planes of the system shall be the ordinary euclidean planes, with the restriction that they shall be determined in the following manner. In the figure here given, let A, B, C be three non-collinear points of the plane α , which we will for convenience take as the plane of the paper.

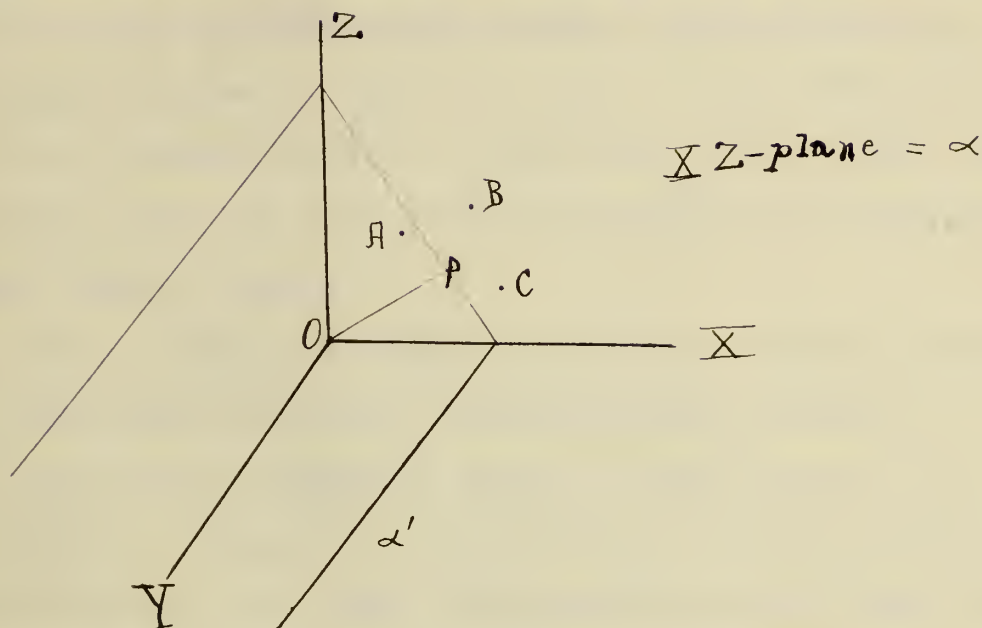


Fig. 8.

Then A, B, C will determine uniquely some circle. Let P be the center of this circle. Then we join PO , O being the origin. Then the plane determined by A, B, C shall be the plane which has as its normal the segment OP . Let us call this plane α' . We see at once that α' is an entirely different plane from the plane α ; and hence I,4 is not fulfilled, for we have three non-collinear points of a plane determining a plane different from the plane in which they lie. In case O and P coincide, then I will define the plane determined by the three non-collinear points as the plane in which they lie. Hence in every

The first part of the paper is devoted to the study of the
 properties of the function $f(x)$ defined by the equation

$$f(x) = \int_0^x \frac{1}{1+t^2} dt$$
 and to the proof of the following theorem:
 The function $f(x)$ is continuous and strictly increasing on the interval $(-\infty, \infty)$.
 The second part of the paper is devoted to the study of the
 properties of the function $g(x)$ defined by the equation

$$g(x) = \int_0^x \frac{1}{1+t^4} dt$$
 and to the proof of the following theorem:
 The function $g(x)$ is continuous and strictly increasing on the interval $(-\infty, \infty)$.

The third part of the paper is devoted to the study of the
 properties of the function $h(x)$ defined by the equation

$$h(x) = \int_0^x \frac{1}{1+t^6} dt$$
 and to the proof of the following theorem:
 The function $h(x)$ is continuous and strictly increasing on the interval $(-\infty, \infty)$.
 The fourth part of the paper is devoted to the study of the
 properties of the function $k(x)$ defined by the equation

$$k(x) = \int_0^x \frac{1}{1+t^8} dt$$
 and to the proof of the following theorem:
 The function $k(x)$ is continuous and strictly increasing on the interval $(-\infty, \infty)$.
 The fifth part of the paper is devoted to the study of the
 properties of the function $l(x)$ defined by the equation

$$l(x) = \int_0^x \frac{1}{1+t^{10}} dt$$
 and to the proof of the following theorem:
 The function $l(x)$ is continuous and strictly increasing on the interval $(-\infty, \infty)$.

case three non-collinear points determine a plane, and axiom I,3 holds. As our straight lines are the ordinary straight lines the plane axioms I,1,2,7, are known to be fulfilled from the reasoning used in paragraph (c). I,5 is fulfilled because our straight lines and planes are of the ordinary kind, and hence, if two points of a line lie in a plane, every point of the line lies in the plane. I,6 is valid since planes, having one point in common have a whole line in common. Axiom I,7 is true because our planes are ordinary and each must have at least three non-collinear points. The space of this geometry, being the same as our ordinary notion of space, must have at least four non-coplanar points. Hence we have shown that axiom I,4 is independent of the other axioms of group I.

This axiom is one which both Schur and Moore claim is deducible from the other axioms of group I and from axioms of group II. In the system we just gave, all axioms of group II hold as well as the other axioms of group I. Axioms II,1-4, which have to do with sequence of points on a line are true since our lines are of the common sort. Axiom II,5 holds for three non-collinear points A,B,C, when joined in pairs, form the sides of the triangle ABC. Then a line, cutting AB internally, will pass through an internal point of AC or BC, provided the line does not pass through A,B, or C. Therefore we have I,4 independent of the remaining axioms of group I and of group II. As before stated, the papers of Schur and Moore will be considered in chapter III.

e) Axiom I,5 is not a consequence of the other axioms of group I.

PROOF. In order to show this fact, we build up a system of geometry, the points of which shall be the points of ordinary geometry, the

straight lines shall be the ordinary straight lines. The planes shall be the common planes with the exception that the points within a small sphere about the origin shall be excluded from our planes. It then follows that all lines passing through this sphere will have certain points which are not points of the plane in which are not points of the plane in which these lines have two points in common. As shown in paragraphs (c) and (d), the plane axioms of the group hold, since points and straight lines are of the ordinary kind. In every case the non-collinear points determine a plane, and also three non-collinear points of a plane determine that plane. Hence axioms I,3,4 are fulfilled. Since two planes, having one point in common, will have at least the major part of a line in common, we have I,6 remaining valid. Axiom I,7 holds since this geometry does no violence to our notions of planes and space as regards the number of independent points existing in each. Therefore we have shown that axiom I,5 is not a consequence of the remaining axioms of group I.

f) Axiom I,6 is independent of the remaining axioms of group I.

PROOF. To demonstrate this, we will restrict our geometry to a cube, inside of which our points, straight lines, planes are of the ordinary kind and are determined in the usual way. Then I,6 does not remain true, for two planes α, β may meet in a single point in any edge of the restricting cube and yet not have a second point in common. In the figure

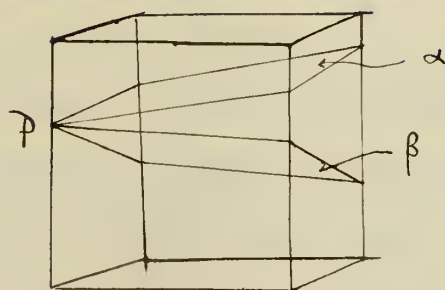
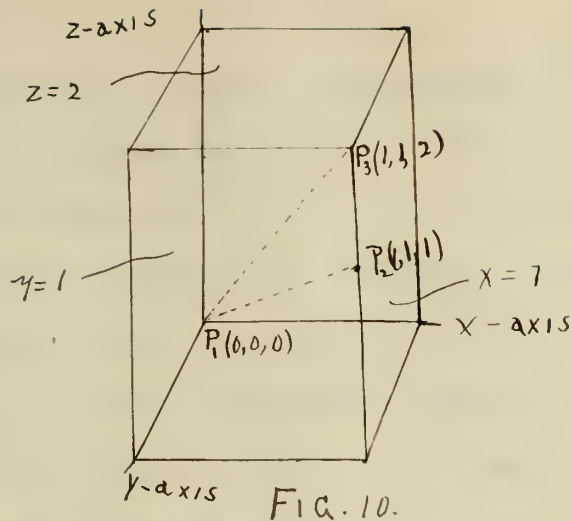


Fig. 9.

here given, the planes α and β have the point P in common, but cannot have a second point of our system in common. Since each line of this system is a part of some ordinary line and by definition the lines are determined in the ordinary manner, we have the plane axioms holding. In other words, any two points of our system determine a line; any two points of a line determine that line, and on every line there exists at least two points. Similarly three non-collinear points determine a plane and the plane determined by them is the plane in which they lie. Hence axioms I, 3, 4 hold. If two points of a line of this system lie in a plane, then the entire line lies in the plane and we have I, 5 holding. The axiom of existence, I, 7, also holds, for every plane must have at least three non-collinear points and space has at least four non-coplanar points. Therefore we have established the independence of axiom I, 6 from the remaining axioms of group I.

g) The independence of axiom I, 7.

PROOF. We must have here a system of geometry which has only one point on a line, two points on a plane, and only three points in space. This geometry must necessarily be very much restricted. In the system which we use for this purpose, our points shall be only those points whose coördinates are positive numbers greater than zero, together with the origin. We shall, furthermore, confine our geometry to the space between the coördinate planes and the planes $x=1, y=1, z=2$. The origin shall not be a point of any line. The planes of the system shall be (1) $x=1$, (2) $y=1$, (3) $x=y$. The lines of the system shall be restricted to (1) $x=y=1$, (2) $x=y=z$, (3) $2x=2y=z$. In this system I, 7 does not hold; for we have the lines $x=y=z$, and $2x=2y=z$, lines which have only one point each.



The planes $x=1$ and $y=1$, furthermore, have each only two points, and in the space of this geometry there are but three points $P_1(0,0,0)$, $P_2(1,1,1)$ and $P_3(1,1,2)$. The other axioms of the group, however, remain valid. I,1 is fulfilled, for the three points of the system, taken in pairs, determine the three lines of the system—I,2 holds; for the only line of our system having two points, namely $x=y=1$, is uniquely determined by the two points $P(1,1,1)$ and $P(1,1,2)$. I,3 is fulfilled, for the three points of the system determine the plane $x=y$. We have, also, the three points $P(0,0,0)$, $P(1,1,1)$, and $P(1,1,2)$ of the plane $x=y$ determining that plane, and hence I,4 is satisfied. Every point of each line lies in the plane of which the line in question is a line, and hence I,5 is fulfilled. I,6 is valid, because all of three planes of the system have two points in common. Therefore we have our proposition.

#3. Independence of the Axioms within Group II.

In this section, I am going to define by suitable conventions the elements of geometry and the word "between" in such a manner that the particular axiom of the group which we are investigating breaks down, while the remaining axioms remain valid. Axioms II,1-4 are linear

axioms, and II,5 is a plane axiom.

a) Axiom II,1 is not a consequence of II,2-4.

PROOF. In the system of geometry which we shall use in this demonstration, our points on a line shall be numbers $\alpha, \beta, \gamma, \dots$. We shall define the idea "between" as follows:- Let α, β, γ be three points on a line. Then β shall lie between α and γ , if β is greater the first point α and less than the second point γ . Then β lies between α and γ , but cannot lie between γ and α ; for from the first condition, $\alpha < \beta < \gamma$. If β could lie between γ and α , we should have $\gamma < \beta < \alpha$, which leads to a contradiction. Therefore β cannot lie between γ and α in the sense in which we have defined "between". Hence II,1 does not hold in this system; for, if B lies between A and C, it cannot lie between C and A. In this system there always exists a point β which lies between α and γ , and a point δ such that γ lies between α and δ , because of the continuity of our number system. Therefore II,2 remains valid. Since only one of three points α, β, γ can lie between the other two points, axiom II,3 is satisfied. Since we can arrange any four points of a line $\alpha, \beta, \gamma, \delta$ so that β shall lie between α and γ and between α and δ , also γ shall lie between α and δ and between β and δ , it then follows that II,4 is fulfilled by this system of geometry. Therefore we have by this system of geometry shown the independence of axiom II,1.

This same system of geometry may be exhibited in the following geometrical manner. The lines of our system shall be positive half-rays emanating from the origin. We shall define "between" to mean that a point B is between the points A and C, if in going along the line positively we pass the three points in the order A, B, C. Here B cannot lie between C and A because of the direction of motion. Again we have the axioms II,2-4 holding, since this system is merely a geom-

metrical interpretation of the number system used in the preceding proof. Hence we have by a second method shown that II,1 is not a consequence of axioms II,2-4.

b) II,2 is independent of the other linear axioms of group II.

PROOF. Let the lines of this geometry be lines of the part of the ordinary plane between two internally tangent circles, one point of each line being the point of tangency. Then II,2 does not hold; for, if we take the point of tangency as C and the second point where the line cuts the inner circle as A, then no point B can lie between A and C in this system, nor can any point D lie beyond C from A, as the only

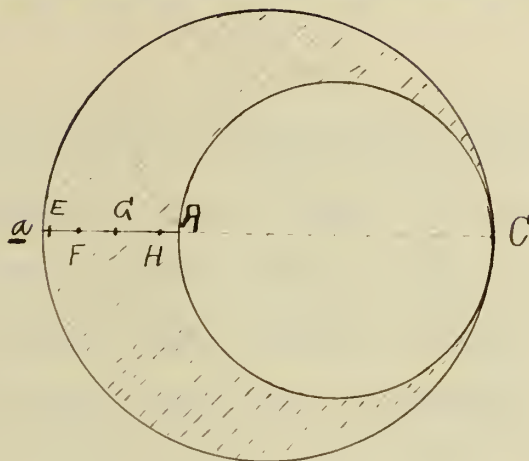


FIG. 11

points of our system are those found in the shaded portion in the figure above. Hence II,2 cannot be fulfilled if we make this choice of A and C. If we take three points, E, F, G of the line a so that F is between E and G, it is also between G and E, and therefore II,1 holds. Also only one of the three points, F is between the other two, and II,3 is likewise fulfilled. Axiom II,4 is also satisfied, since any four points E, F, G, H of the line a may be arranged so that F shall lie between E and G and between ^{and} \overline{EH} , also G shall lie between E and H and F and H. Therefore we have shown that II,2 is independent of axioms II,

1,3,4.

c) Axiom II,3 is not a consequence of axioms II,1,2,4.

PROOF. To show this fact we will build up a geometry in which the points are ordinary points and the straight lines are circles. Then II,3 is not valid for any one of the three points A,B,C of the line a, as shown in the figure, will lie between the remaining two points. Ax-

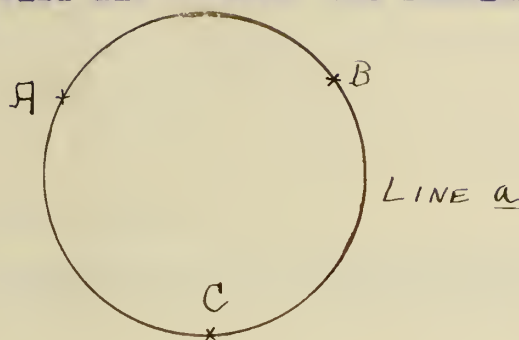


Fig. 12.

iom II,1 is true, for in this geometry the point B, which lies between A and C also lies between C and A. Since the circle is a continuous curve, there always exists on any straight line of this system between the points A and C a point B, and likewise beyond the point C from A there is a point D on the given line. Furthermore, any four points A,B,C,D may be so arranged that B shall lie between A and C and A and D, and also C shall lie between A and D and B and D, which is nothing more than axiom II,4 which is certainly valid in this system. Therefore we have shown that axiom II,3 is independent of the linear axioms of group II.

d) The independence of axiom II,4 from II,1,2,3.

PROOF. A system which Hilbert gave in his lectures during the Winter semester of 1898-'99 is very easily understood. Hence I shall give it at this place. Evidently the system of geometry used to show the independence of this axiom must be some number system, such as Hilbert uses.

Hilbert's proof. Take as the points A,B,C,---of the new geometry the numbers α, β, γ ,---, and suppose that C lies between A and B, providing we have $\gamma > \alpha$ and $\gamma > \beta$. It is evident then that axioms II,1,2,3 are all valid. The axiom II,4 is, however, not valid; for let A,B,C,D be a sequence of four points in the meaning of axiom II,4. Then we must have

$$\beta > \alpha, \beta > \gamma, \beta > \delta$$

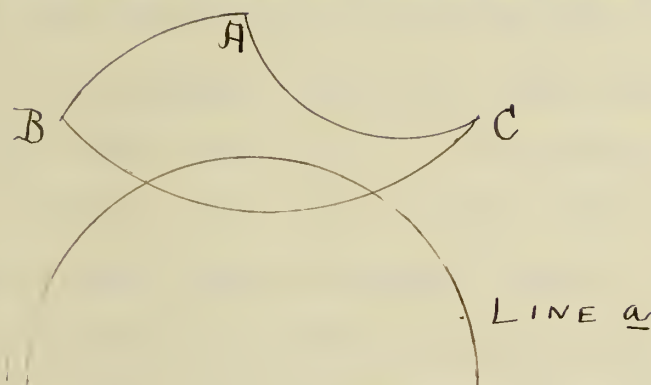
and also

$$\gamma > \alpha, \gamma > \beta, \gamma > \delta.$$

But these two sets of inequalities are inconsistent, and hence our proposition follows.

e) Axiom II,5 is independent of all the other axioms of group II.

PROOF. In this new geometry the points shall be the common points, and the straight lines shall be semi-circles. Segments of straight lines shall be circular arcs less than a semi-circle. Then II,5 is not fulfilled; for, in this system, a line may cut one and only one of the segments AB, AC, BC. In the figure here given, we have the line a cut-



ting only the segment BC. The sequence of the points on such lines as a according to axioms II,1-4 remains entirely valid. Hence our proposition follows. As a second method of proof, we may use the fol-

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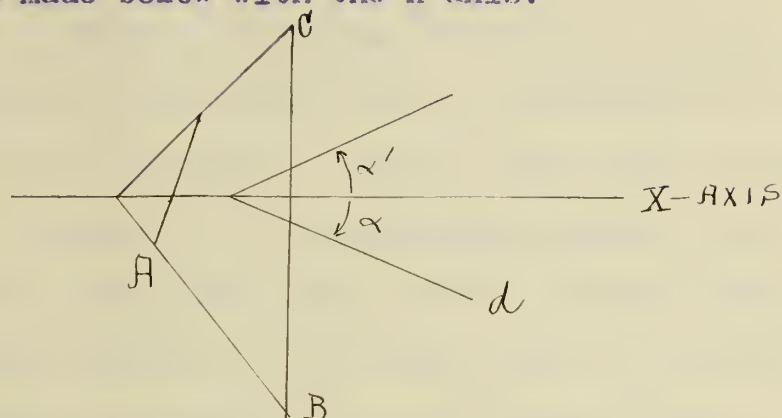
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lowing system of geometry. * In this system of geometry points are the ordinary points. The straight lines are the ordinary straight lines, with the exception that lines cutting the axis of x from below at an angle less than 90° , shall change their direction at the point of crossing so that the angle made with the axis of x above shall be the same as the angle made below with the x -axis.



For example, in the figure here given, the line d makes the acute angle α with the x -axis from below and is deflected on crossing the axis of x so that the angle α' is equal to α . Then II,5 does not hold as shown in the above figure, for the line d cuts only the segment BC of the sides of triangle ABC. Since points on the straight lines of the system are arranged in the ordinary manner, we shall have as a relation of the points A, B, C, that the point B lying between A and C and also between C and A. On any line of this system there always exists a point between A and C and a point D beyond C from A. Of three points A, B, C, only one can lie between the other two points. Furthermore, any four points A, B, C, D of a line can be arranged according to axiom II,4. Hence we have all of the axioms II,1-4 holding. Therefore we have a-

* This system is somewhat like the non-desarguesian geometry given by

Dr. F. R. Moulton before a recent meeting of the American Mathematical Society (See Transactions of American Mathematical Society, April, 1902) My own system was, however, arrived at independently.

gain shown that II,5 is not a consequence of the other axioms of group II.

This completes the axioms of group II.

4. The Independence within the group of the Axioms of Group IV.

In this section, I consider the axioms of Hilbert's fourth group and show that each axiom of the group is not a consequence of the others of the group. In each system of geometry here used, the things to be defined are the elements of the particular geometry and the notion "congruent". Where points and lines are not defined, they are supposed to be used in the ordinary or euclidean sense. Of this group, axioms IV,1-3 are linear axioms, and IV,4-6 are plane axioms.

a) IV,1 is independent of axioms IV,2,3, the linear axioms of the group.

PROOF. In this geometry, the length of each segment shall be equal to a constant value \underline{m} . Segments having the same length are defined as congruent, and hence, in this system of geometry, every segment is congruent with every other segment of the system. Then IV,1 is not fulfilled; for, any number of segments of the line \underline{a} on the same side of the point A may be found such that they will be congruent to a given

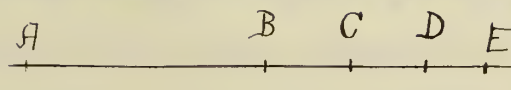


Fig. 15

segment AB. In the figure, we have $AC=AB$, $AD=AB$, etc. Axiom IV,2 holds, since every segment in the geometry is congruent with every other segment of the system. IV,3 is valid, since the length of a segment, which is the sum of two segments, is in turn equal to \underline{m} , as is also a second

segment similarly related to two segments. Hence we have established our proposition.

b) Axiom IV,2 is not a consequence of axioms IV,1,3.

PROOF. In this new geometry, we will define two line-segments as congruent, if the length of the second segment multiplied by the cosine of the angle between the two segments, is equal to the length of the first segment. Then axiom IV,2 is not satisfied.

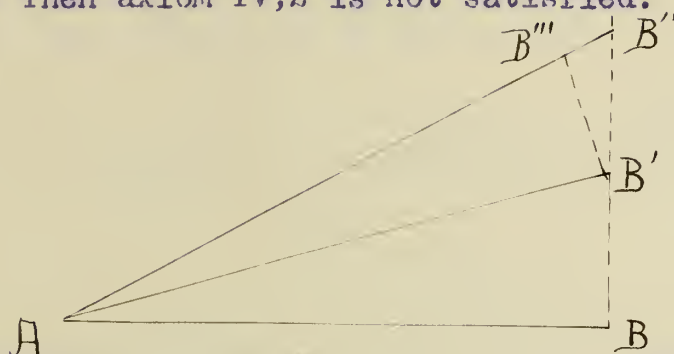


FIG. 16.

In the above figure, we have the congruences

$$AB \equiv AB'$$

and $AB \equiv AB''$

But $AB' \not\equiv AB''$, where $B' \neq B''$.

Therefore $AB' \not\equiv AB''$, and IV,2 is not valid. At the same time, axiom IV,1 is satisfied; for, on a given side of a point A on a line a, only one segment can be found such that it shall be congruent with a given segment. Furthermore, the sum of two segments, AB and BC, of a straight line is congruent with the sum of the segments, A'B' and B'C', of a second line, if $AB \equiv A'B'$ and $BC \equiv B'C'$. Consequently IV,3 holds. Hence the independence of axiom IV,2 from IV,1,3 follows.

c) Axiom IV,3 is independent of IV,1,2.

PROOF. In this system of geometry, we will define the length of a segment as follows:-

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Let $L = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$, where $|y_1| < |y_2|$.

Segments of equal length in this sense shall be congruent. Then IV,3 does not hold, as we shall see in the following case.

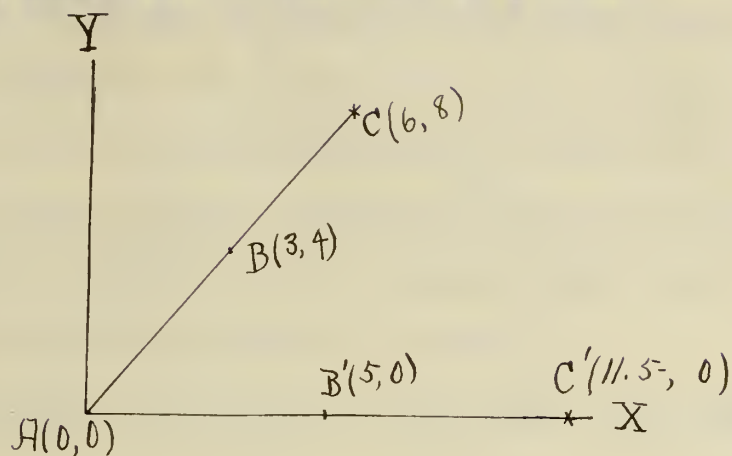


FIG. 17.

Let the extremities of AB be (0,0) and (3,4). Then $AB=5$, and on the x-axis we lay off $AB'=5$. Let C be (6,8). Then we have the length of $BC = \sqrt{41} = 6.5$ in this system of geometry. Then on the x-axis lay off $B'C' = 6.5$. Hence we have $AB \equiv AB'$ and $BC \equiv B'C'$. But $AC=10$, while $AC' = 11.5$, and therefore $AC \neq AC'$. So IV,3 does not hold. At the same time IV,1 holds, since on a given side of a point on a given line only one segment can be congruent with a given segment. Axiom IV,2 is also valid; for, if two segments are congruent with a given segment of a line, they are congruent with each other. Hence we have shown that IV,3 is independent of axioms IV,1,2.

d) Axiom IV,4 is not a consequence of the other axioms of group IV.

PROOF. In the system of geometry that we use for this purpose, our angles shall be congruent, if their sines are equal. All triangles of this system shall be acute-angled. Lines shall have their segments congruent in the ordinary sense. Then IV,4 breaks down; for, we see

1. The first part of the paper is devoted to the study of the

properties of the function $f(x)$ defined by the equation

$$f(x) = \int_0^x \frac{1}{1+t^2} dt, \quad x \in \mathbb{R}.$$

It is shown that $f(x)$ is a strictly increasing function and

that $f(x)$ is bounded on \mathbb{R} . The limits of $f(x)$ as $x \rightarrow \pm\infty$

are found. The second part of the paper is devoted to the study of

the function $g(x)$ defined by the equation

$$g(x) = \int_0^x \frac{t}{1+t^2} dt, \quad x \in \mathbb{R}.$$

It is shown that $g(x)$ is an odd function and that $g(x)$ is

bounded on \mathbb{R} . The limits of $g(x)$ as $x \rightarrow \pm\infty$ are found.

The third part of the paper is devoted to the study of the

function $h(x)$ defined by the equation

$$h(x) = \int_0^x \frac{1}{1+t^4} dt, \quad x \in \mathbb{R}.$$

It is shown that $h(x)$ is a strictly increasing function and

that $h(x)$ is bounded on \mathbb{R} . The limits of $h(x)$ as $x \rightarrow \pm\infty$

are found. The fourth part of the paper is devoted to the study of

the function $k(x)$ defined by the equation

$$k(x) = \int_0^x \frac{t^2}{1+t^4} dt, \quad x \in \mathbb{R}.$$

It is shown that $k(x)$ is an even function and that $k(x)$ is

bounded on \mathbb{R} . The limits of $k(x)$ as $x \rightarrow \pm\infty$ are found.

The fifth part of the paper is devoted to the study of the

function $l(x)$ defined by the equation

$$l(x) = \int_0^x \frac{1}{1+t^6} dt, \quad x \in \mathbb{R}.$$

It is shown that $l(x)$ is a strictly increasing function and

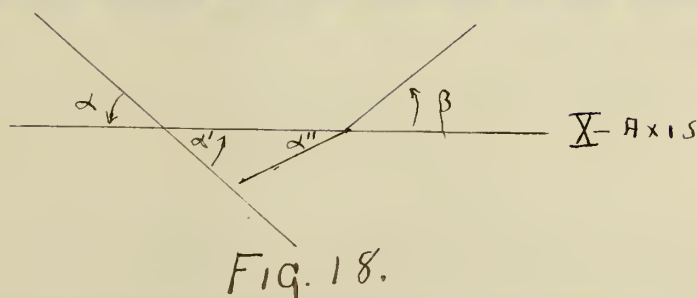
that $l(x)$ is bounded on \mathbb{R} . The limits of $l(x)$ as $x \rightarrow \pm\infty$

are found. The sixth part of the paper is devoted to the study of

at once that on a given side of a half-ray we may have two angles which will be congruent with a given angle. In this system, the linear axioms hold by definition of linear congruence. Axiom IV,5 is valid; because, if we have two angles congruent with a given angle, they are congruent with each other. The sines of all three angles will be the same. Furthermore, IV,6 is satisfied, since the angles of every triangle are acute, and hence we will have the ordinary congruence of the angles of triangles. Therefore we have demonstrated that IV,4 is not dependent upon the remaining axioms of group IV.

e) The independence of axiom IV,5 from the other axioms of group IV.

PROOF. We will build up a system of geometry in which segments of lines are congruent in the ordinary sense. The straight lines shall be the ordinary straight lines with the restriction, however, that lines, making with the x -axis from above a positive angle less than a right angle, shall have their direction changed on crossing the x -axis so that the positive angle the line makes with the x -axis below shall have its tangent equal to the sine of the angle made above. Angles in each region (above or below the x -axis) shall be congruent with an angle of the same region in the ordinary sense. Furthermore, the theorem of equality of vertical angles shall hold. All the triangles of the system shall be entirely above or below the axis of x . Then axiom IV,5 is not valid in this system as is shown below. In the



It is not clear if a letter from the author of the book is included in the book. The book is a collection of letters from the author to his friends and family. The letters are written in a very personal and intimate style. The author is a very good writer and his letters are full of life and interest. The book is a very good read and it is a pleasure to read the letters. The letters are written in a very personal and intimate style. The author is a very good writer and his letters are full of life and interest. The book is a very good read and it is a pleasure to read the letters.

1. The Introduction

The book is a collection of letters from the author to his friends and family. The letters are written in a very personal and intimate style. The author is a very good writer and his letters are full of life and interest. The book is a very good read and it is a pleasure to read the letters. The letters are written in a very personal and intimate style. The author is a very good writer and his letters are full of life and interest. The book is a very good read and it is a pleasure to read the letters. The letters are written in a very personal and intimate style. The author is a very good writer and his letters are full of life and interest. The book is a very good read and it is a pleasure to read the letters.

figure here given, let $\angle \beta \equiv \angle \alpha$. From our definitions of congruence we have $\alpha \equiv \alpha'$, and $\angle \beta \equiv \angle \alpha \equiv \angle \alpha''$. But α' cannot be congruent with α'' , since they are angles in the same region and the tangent of α'' is equal to the sine of α' . Hence IV,5 fails. Since the congruence of segments is used in the ordinary sense, the linear axioms of the group, IV,1-3, remain valid. And likewise, since on a given side of a half-ray only one angle can be constructed congruent with a given angle, we have IV,4 holding. Because our triangles are restricted either to the region above or the region below the x-axis, where the ordinary relations of congruence are true, axiom IV,6 has its truth undisturbed. Therefore we have established the independence of IV,5 from its group.

f) The independence of IV,6 from the other axioms of group IV.

PROOF. We will construct a new geometry in which we shall let our straight lines be circles, and let segments be some part of a circle. Two segments shall be congruent, if their rectified lengths are equal. The congruence of angles shall be defined in the ordinary manner. Our triangles shall be the plane surface intercepted within three intersecting circles. The sides of the triangle shall be the intercepted arcs. We see at once that axiom IV,6 does not hold. In this figure, we

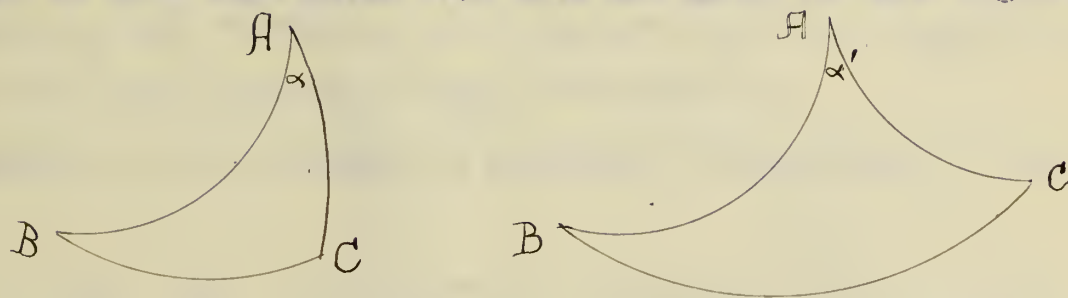


Fig. 19.

have $AB \equiv A'B'$, $AC \equiv A'C'$, and angle $\alpha \equiv \alpha'$. But angle ACB is not congruent with angle $A'C'B'$, and also angle ABC is not congruent with angle $A'B'C'$. These non-congruences cause IV,6 to break down. The linear axioms of the group are satisfied, since rectified arcs will obey the same congruence laws as segments of ordinary straight lines. IV,4 holds, for on a given side of an arc only one second arc can cut the given arc with a given angle. Furthermore, we have IV,5 valid, for, if two angles of the system are congruent with a third angle, they must be congruent with each other. Hence the truth of our proposition follows.

This system completes my proofs for the independence of the axioms of group IV within the group. I have now accomplished what I started out to do, for in each case I have shown that each axiom of Hilbert's groups is not a consequence of the other axioms of its own group.

CHAPTER III.

SOME MATHEMATICAL WORK CLOSELY RELATED to my INVESTIGATIONS.

In this chapter I shall give briefly some of the results obtained by Hilbert in his "Foundations of Geometry", and also a short discussion of the recent papers of Schur * and Moore *.

#1 Summary of the remainder of Hilbert's "Foundations of Geometry."

In the proofs that I formulated in the second chapter of this thesis, I considered only the independence of axioms within their individual groups.

*Schur's "Ueber die Grundlagen der Geometrie" (Mathematische Annalen, 1901, pp. 265-292 and Moore's "On Projective Axioms" (Transactions of the American Mathematical Society, Vol. 3, No. 1, pp. 142-158, January, 1902).

We see at once that the greater the number of axioms regarded as valid the more complex will become the system of geometry, used to show the independence of a certain axiom.

Hilbert makes the assertion that "the properties of euclidean space are used merely as abbreviated notations of certain arithmetical relations." Accordingly in developing certain geometrical propositions, Hilbert makes use of the following properties of number systems:-

Theorems of connection (1-12).

1. From the number a and the number b, there is obtained by "addition" a definite number c, which we express by writing

$$\underline{a} + \underline{b} = \underline{c} \quad \text{or} \quad \underline{c} = \underline{a} + \underline{b}$$

2. There exists a definite number, which we call 0, such that, for every number a, we have

$$\underline{a} + 0 = \underline{a} \quad \text{and} \quad 0 + \underline{a} = \underline{a}.$$

3. If a and b are two given numbers, there exists one and only one number x, and also one and only one number y, such that we have respectively,

$$\underline{a} + \underline{x} = \underline{b}, \quad \underline{y} + \underline{a} = \underline{b}.$$

4. From the number a and the number b, there can be obtained in another way, namely by "multiplication", a definite number c, which we express by writing

$$\underline{a}\underline{b} = \underline{c} \quad \text{or} \quad \underline{c} = \underline{a}\underline{b}.$$

5. There exists a definite number, called 1, such that for every number a, we have

$$\underline{a} \cdot 1 = \underline{a} \quad \text{and} \quad 1 \cdot \underline{a} = \underline{a}.$$

6. If a and b are any arbitrarily given numbers, where a is dif-

we see at once that the function f is continuous at a and that $f(a) = L$. In the same manner we shall show that f is continuous at b and that $f(b) = L$. The theorem is proved.

It is now easy to see that the function f is continuous at every point x in the interval (a, b) . For if x is any point in this interval, then x is not equal to a or b , and hence f is continuous at x . This completes the proof of the theorem.

THEOREM 11-11

1. If f is a function defined on the interval $[a, b]$ and if f is continuous at every point in the interval (a, b) , then f is continuous at a and b .

$$f(a) = L \text{ and } f(b) = L$$

2. If f is a function defined on the interval $[a, b]$ and if f is continuous at every point in the interval (a, b) , then f is continuous at a and b .

$$f(a) = L \text{ and } f(b) = L$$

3. If f is a function defined on the interval $[a, b]$ and if f is continuous at every point in the interval (a, b) , then f is continuous at a and b .

$$f(a) = L \text{ and } f(b) = L$$

4. If f is a function defined on the interval $[a, b]$ and if f is continuous at every point in the interval (a, b) , then f is continuous at a and b .

$$f(a) = L \text{ and } f(b) = L$$

5. If f is a function defined on the interval $[a, b]$ and if f is continuous at every point in the interval (a, b) , then f is continuous at a and b .

$$f(a) = L \text{ and } f(b) = L$$

6. If f is a function defined on the interval $[a, b]$ and if f is continuous at every point in the interval (a, b) , then f is continuous at a and b .

ferent from 0, then there exists one and only one number \underline{x} and also one and only one number \underline{y} such that we have, respectively,

$$\underline{ax} = \underline{b} , \underline{ya} = \underline{b}.$$

If $\underline{a}, \underline{b}, \underline{c}$ are arbitrary numbers these rules for calculation always hold:-

$$7. \underline{a} + (\underline{b} + \underline{c}) = (\underline{a} + \underline{b}) + \underline{c}.$$

$$8. \underline{a} + \underline{b} = \underline{b} + \underline{a}.$$

$$9. \underline{a}(\underline{bc}) = (\underline{ab}) \underline{c}.$$

$$10. \underline{a}(\underline{b} + \underline{c}) = \underline{ab} + \underline{bc}.$$

$$11. (\underline{a} + \underline{b}) \underline{c} = \underline{ac} + \underline{bc}.$$

$$12. \underline{ab} = \underline{ba}.$$

Theorems of order (13-16).

13. If $\underline{a}, \underline{b}$ are any two distinct numbers, one of these, say \underline{a} , is always greater ($>$) than the other. The other number is said to be the smaller of the two. We express this relation by writing

$$\underline{a} > \underline{b} \text{ and } \underline{b} < \underline{a}.$$

$$14. \text{ If } \underline{a} > \underline{b} \text{ and } \underline{b} > \underline{c}, \text{ then is also } \underline{a} > \underline{c}.$$

$$15. \text{ If } \underline{a} > \underline{b}, \text{ then is also } \underline{a} + \underline{c} > \underline{b} + \underline{c} \text{ and } \underline{c} + \underline{a} > \underline{c} + \underline{b}.$$

$$16. \text{ If } \underline{a} > \underline{b} \text{ and } \underline{c} > 0, \text{ then is also } \underline{ac} > \underline{bc} \text{ and } \underline{ca} > \underline{cb}.$$

Theorem of Archimedes.

17. If $\underline{a}, \underline{b}$ are any two arbitrary numbers, such that $\underline{a} > 0$ and $\underline{b} > 0$, it is always possible to add \underline{a} to itself a sufficient number of times so that the resulting sum shall have the property that

$$\underline{a} + \underline{a} + \underline{a} + \text{-----} + \underline{a} > \underline{b}.$$

A number system which possesses only a part of these properties (1-17) is called a complex number system. A number system is called archimedean or non-archimedean according as it does or does not sat-

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isfy condition 17.

By the aid of axioms I,1-2 and II-IV and by the use of the above properties of numbers (with the exception of 17),Hilbert develops Euclid's theory of proportion.

At this point he gives a proof of Pascal's theorem,since it has much to do with his subsequent work. This theorem is stated as follows:-

PASCAL'S THEOREM:- Given two sets of points A,B,C and A',B',C' so situated respectively upon two intersecting straight lines that none of them fall at the intersection of these lines. If CB' is parallel to BC' and CA' is also parallel to AC' ,then BA' is parallel to AB' (See figure here given).

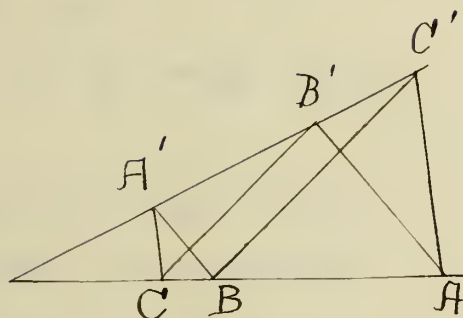


FIG. 20.

Hilbert gives two proofs for this theorem. In the first he makes use of axioms I 1-2,II-IV. In his second proof some of the axioms of congruence are replaced by one for the isosceles triangle,otherwise the proof being the same.

Upon the theorem of Pascal,Hilbert bases an algebra of segments. The sum of two segments on the same straight line is defined in the usual way. He defines the product of two segments as follows:- On one side of a right angle lay off from the vertex 0 the segment a on the other side the segments l and b; then draw l a and through b the

parallel to l a; this parallel will cut off on the other side a seg-

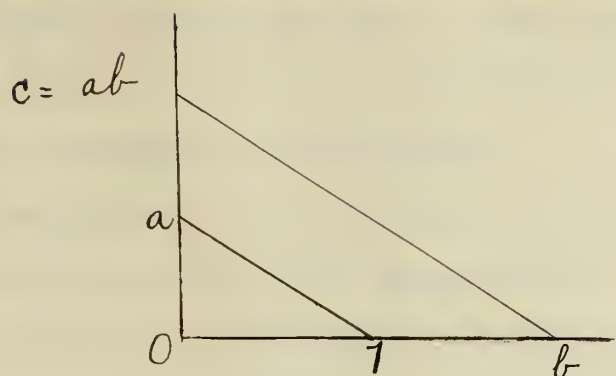


FIG. 21.

ment c (counted from 0) which is defined as the product $\underline{c} = \underline{a}\underline{b}$ of the segment a into the segment b. With this convention, Hilbert shows by Pascal's theorem that the commutative law

$$(\underline{a}\underline{b} = \underline{b}\underline{a})$$

and the distributive law

$$(\underline{a}(\underline{b} + \underline{c}) = \underline{a}\underline{b} + \underline{a}\underline{c})$$

are valid.

One sees at once the close relation of this algebra to the theory of proportion. Take the proportion

$$a:b = a' : b',$$

where a, b, a', b' are any segments and let this proportion be defined as equivalent to the equation

$$a \cdot b' = a' \cdot b.$$

Then the theory of proportion evidently holds.

By the aid of similar triangles defined in the usual way, Hilbert shows that a straight line is represented by a linear equation. He also uses Pascal's theorem as the basis for the theory of areas of plane figures. Hilbert defines two polygons as having equal areas, if they can be resolved into a finite number of triangles that are congruent in pairs. When to each of two polygons it is possible to

add polygons of equal area so that the resulting polygons have equal areas, the two given polygons are said to have equal content. These definitions are distinct, since the investigations are carried on independent of the principle of Archimedes.

As a neat application of his results in this line, Hilbert demonstrates in a new way the following proposition:-

If, after decomposing a rectangle by means of straight lines into a number of triangles, any one of these triangles be omitted, it will be impossible to make up the given rectangle from the remaining triangles.

Hilbert next considers the theorem of Desargues, which is stated in the following manner:-

Theorem of Desargues. When two triangles are so situated in a plane that their homologous sides are respectively parallel, then the lines joining the homologous vertices pass through one and the same point, or are parallel to each other.

This theorem is an immediate consequence of axioms I-III. However, if I, 3-7, which are space axioms, are not used, it is impossible to prove Desargues's proposition, and hence we see without any difficulty that the theorem of Desargues is a necessary condition that the plane geometry become a part of the geometry of space. Hilbert shows further that not all linear and plane axioms hold, if we have a system of geometry in which Desargues's theorem is not valid. If, in the algebra of segments, we discard this theorem, the theorems of connection hold with the exception of the commutation law

$$\underline{ab} = \underline{ba}.$$

This fact shows another important use of the theorem in question. The number system in which $\underline{ab} \neq \underline{ba}$ is called a desarguesian number sys-

tem.

In this section, I have summed up very briefly Hilbert's main results depending upon the important theorems of Pascal and Desargues. For complete detail the reader is referred to Prof. E.J. Townsend's translation of Hilbert's "Grundlagen der Geometrie."

#2. An Outline of Schur's Article "Ueber die Grundlagen der Geometrie" (Mathematische Annalen, 1901, pp. 265-292)

Schur takes Hilbert's axioms of connection and order and classifies them as projective axioms. Schur claims for his system that it includes all that Hilbert gives in I-II. Schur's axioms, or as he calls them postulates, are stated as follows:-

Postulate 1. There exists an unlimited number of elements, which we call points.

Postulate 2. Any two distinct points determine uniquely an assemblage of an unlimited number of points, to which they themselves belong. This assemblage of points he calls a "segment". Any two points of a segment determine another segment whose points belong to the first segment.

Postulate 3. Every segment AB determines two other classes of points, namely, its continuations beyond B and A. These two classes have the property that every point of the first determines with A a segment to which B belongs and every point of the second determines with B a segment to which A belongs. If C is a point of the segment AB, the continuation of CB beyond B will coincide with the continuation of AB beyond B. Furthermore, the continuation of CA beyond B will consist of AC and its continuation beyond B.

Postulate 4. There exists no points which belong simultaneously

to both continuations of a segment.

He here defines a line as consisting of all points in both of the continuations of a segment.

Postulate 5. There exist points outside of each line.

Postualte 6. If A,B,C are three non-collinear points, and if D is a point of the segment BC and, further, if E is a point of the segment AD, then there will exist a point F of the segment AB, such that E will lie on the segment CF.

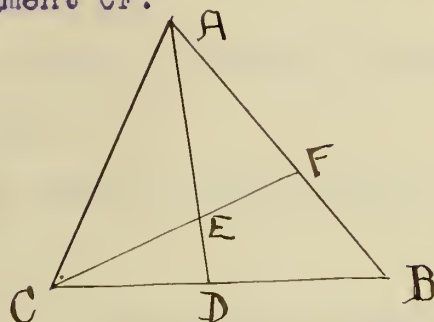


FIG. 22.

Postulate 7. If A,B,C are not points lying in the same straight line, and D is a point of the segment BC and F is a point of AB, then there will exist a point E, which is common to the segments AD and CF.

Schur defines a plane as follows:-

The assemblage of the points of those segments (straight lines), which join each of three non-collinear points with the points of the segment determined by the other two points, is called a triangle (a plane).

Postulate 8. There exist points outside of each plane.

Space is defined as an assemblance of points of all the lines of two classes. The first class consists of those lines which join each of four non-coplanar points with the points of the triangle determined by the other three points. The second class of lines consists of those lines which join the points of the line determined by

any two of the four non-coplanar ^{points} with the points of the line determined by the other two points. Such an assemblage of points is a space.

Postulate 9. There exists no points outside of space.

I give below some of Schur's theorems resulting from his postulates.

Theorem 1. A straight line is uniquely determined by any two of its points.

Theorem 2. A plane is entirely determined by any three of its non-collinear points.

Theorem 4. A space is determined by any four of its non-coplanar points.

Theorem 6. Two planes of the same space, having one point in common, will also have a line in common.

Theorem 1 is identical with Hilbert's axiom I,2, theorem 4 is the same as Hilbert's I,4, and theorem 6 is likewise equivalent to Hilbert's axiom I,6. Schur includes in the definition of his plane the substance of Hilbert's axiom I,3. As before stated, Schur maintains that his system of postulates includes all given in Hilbert's groups of axioms I-II. If this be true, then we have at once that Hilbert's axioms I,3-5 are redundancies. This fact is what Schur attempts to prove.

Schur's fallacy, however, arises from the fact that his linear segments are not the same as the ordinary straight line, nor are his planar segments (triangles) the equivalent of the euclidean planes which Hilbert considers in his system of axioms. Schur insists upon calling line and linear segment the same, as also he interchanges indiscriminately the triangle and plane. Because of this fact, Schur's

entire criticism of Hilbert's system of axioms appears to break down.

Professor Moore, in the paper which I next discuss, builds up a system of geometry to show that Schur's results concerning I, 3-5 are not correct. In this system of geometry he lets the two planes of his system be the surfaces of two intersecting spheres. Points and straight lines are of the ordinary kind. In this system Hilbert's axioms I, 1, 2, 6, 7 and II, 1-5 are valid, while I, 3-5 do not hold. I, 3 fails because three non-collinear points will not determine one of our planes; and I, 4 is not valid for three non-collinear points of either plane will not determine that plane, since four points are necessary to determine the surface of a sphere. I, 5 is not true for a straight line will pierce the surface of a sphere in two points and yet have an infinity of points which are not points of the surface of the sphere (his plane). This system shows at once that axioms I, 3-5 are not deducible from the other axioms of group I and II.

In the latter part of Schur's paper, he takes up the axioms of congruence, Pascal's theorem, and the arithmetic of projective segments on much the same line of discussion ^{as Hilbert's discussion} of the same subjects. These subjects, however, do not bear directly on my work, and hence they are allowed to pass with mere mention.

#3. MOORE'S PAPER on PROJECTIVE AXIOMS.*

In #1 of Moore's paper on Projective Axioms of Geometry, he formulates a set of seven axioms which are generalized for n-dimensional geometry ($n \geq 2$). For $n=3$, Moore's axioms correspond with Hilbert's axioms as follows:-

$$(1; 2; 3; 4; 5; 6; 7) = (I, 1, 2, 7; II, 1; II, 3; II, 5; II, 2; I, 7_1, I, 7_2).$$

* Transactions of the American Mathematical Society, Vol. 3, No. 1, pp. 142-158, January, 1902.

THE SECRETARY OF THE ARMY AND NAVAL DEPARTMENT
WASHINGTON, D. C.
JANUARY 10, 1917
SIR:
I have the honor to acknowledge the receipt of your letter of the 9th inst. in relation to the subject of the proposed amendment to the regulations governing the appointment of officers to the grade of Captain in the United States Army.
The proposed amendment is being considered by the War Department and it is expected that a decision will be reached in the near future.
Very respectfully,
THE SECRETARY OF THE ARMY AND NAVAL DEPARTMENT

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TO THE SECRETARY OF THE ARMY AND NAVAL DEPARTMENT
WASHINGTON, D. C.
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In the cases underscored the correspondence is only approximate. In his system, there is nothing corresponding to 1,3,4,5 as the plane (and in general the k -spaces) are explicitly defined.

Moore's groups of projective axioms are as follows:-

1-4; Axioms of conditioned existence and of definition ;

5-6; Axioms of absolute existence;

7; Axiom of limitation.

This division is made on the basis of what Moore takes to be the principal functions of the various axioms in the body of axioms.

This is the list of his axioms:-

Axiom 1; The Line. Two distinct points A,B determine uniquely a set of points, the line AB. The points A,B belong to or lie on the line AB. A line is a set of points thus determined by certain two and indeed by any two distinct points belonging to it.

Axiom 2: The Segment of a Line. Two distinct points A,B of a line determine uniquely a set of points distinct from A,B and lying on the line; this set of points is the segment AB, with extremities A,B. The segment AB contains and thus is identical with the segment BA. A segment is a set of points thus determined by two distinct points, its extremities.

Moore here defines point of a segment in the following manner:- A point C of the segment AB is said to lie between, or to separate , the extremities A,B of the segment, in the notation ACB or BCA. The notation indicates the order-relation of the three points.

Axiom 3: Three Collinear Points. Of three distinct collinear points A,B,C one and only one lies on the segment determined by the other two; of the three order-relations, ABC, BCA, CAB one and only one holds.

Axiom 4: The Triangle with the Transversal Line. A line which cuts

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one side of a triangle externally and another side internally cuts the third side internally.

Then Moore gives definitions of independent points and elements determined by them.

($k=2$). Three distinct points A_1, A_2, B_0 are (in the order specified) independent, if B_0 does not lie in the line A_1, A_2 .

($k=2$). Three independent points A_1, A_2, B_0 determine uniquely a plane, or 2-space, A_1, A_2, B_0 . The plane A_1, A_2, B_0 is the aggregate set of all points A of the line A_1, A_2 , of all points C separated from B_0 by the line A_1, A_2 , and of all points B separated from at least one of these points C by the line A_1, A_2 .

The above two definitions are generalized for $k = 3$ and finally for $k = k$, thus making Moore's axioms a set of axioms of the greatest possible generality.

Axiom 5. On the line AB determined by two distinct points A, B there exists a point C on the segment AB and there exists a point D distinct from A such that the segment AD contains B .

Axiom 6. There exists in the fundamental space a set of $n+1$ independent points A_1, \dots, A_n, A_{n+1} (where n is any particular integer greater than 1); and thus there exists a set of $k + 1$ independent points A_1, \dots, A_k, A_{k+1} for every integer k ($1 \leq k \leq n$).

Axiom 7. There exists in the fundamental space no $n + 1$ - space.

In # 2 Moore gives an indirect proof that Hilbert's axiom II,4 is not independent of groups I and II, by proving that the axiom in question depends as a theorem on axioms 1-6, in particular on axiom 4, the triangle-transversal axiom.

The outline of his proof is the following:-

He states that, "in the usual geometry, one will admit the validity of

The first of the three conditions is that the function f is continuous on the interval $[a, b]$. The second condition is that the function f is bounded on the interval $[a, b]$. The third condition is that the function f is integrable on the interval $[a, b]$.

It is well known that if a function f is continuous on the interval $[a, b]$, then it is also bounded and integrable on the interval $[a, b]$. Therefore, the first two conditions are satisfied.

Now we need to check the third condition, which is that the function f is integrable on the interval $[a, b]$. To do this, we will use the definition of the Riemann integral.

Let P be a partition of the interval $[a, b]$. Then the Riemann sum of the function f over the interval $[a, b]$ with respect to the partition P is given by

$$S(P, f) = \sum_{i=1}^n f(\xi_i) (x_i - x_{i-1})$$

where ξ_i is any point in the subinterval $[x_{i-1}, x_i]$. The function f is said to be integrable on the interval $[a, b]$ if there exists a real number I such that

$$\lim_{\|P\| \rightarrow 0} S(P, f) = I$$

where $\|P\|$ is the norm of the partition P , defined as the maximum length of the subintervals in P . In this case, the limit exists and is equal to I .

Since the function f is continuous on the interval $[a, b]$, it follows that the limit exists and is equal to the definite integral of f over the interval $[a, b]$.

Therefore, the function f is integrable on the interval $[a, b]$, and the third condition is satisfied.

Since all three conditions are satisfied, we conclude that the function f is integrable on the interval $[a, b]$.

The value of the integral is given by the definite integral of f over the interval $[a, b]$.

Therefore, the value of the integral is I .

the following four statements concerning the order-relations of four distinct collinear points A,B,C,D.

1. If CAB and ABD, then CAD and CBD.
2. If CAB and ADE, then CAD and CDB.
3. If CAB and DAB, then either CDA or DCA; either CDB or DCD; if CDA, then CDB, and if CDB, then CDA.
4. If ACB and ADB, then either ACD or ADC; either CDB or DCB; if ACD, then CDB; and if CDB, then ACD.

Moore asserts that these four statements are equivalent to Hilbert's linear order axiom II,4:-

To any four distinct points of a line the notation A,B,C,D may always be assigned in such a way that ABC, ABD, ACD, BCD.

Moore then takes up each of the four statements and proves that each is deducible from axioms 1-6.

Professor Moore devotes section 3 to properties of the k -spaces of the fundamental space. In his system, we find neat generalizations for any dimensional space and the relations existing between lower spaced elements.

His theorem, or group of theorems, giving these properties are stated as follows:-

THEOREM.- In a fundamental space for which the axioms 1-6 with $n \geq 2$ are valid, the sets of independent points and the k -spaces have the following properties:

1. Any two distinct points of a k -space determine a line or 1-space lying entirely in it.
2. A k -space is determined by any set of $k+1$ independent points lying in it.

3. A k -space contains no set of $k+2$ independent points.

4. Any $k'+1$ independent points of a k -space ($0 \leq k' \leq k$) determine a k' -space lying entirely in it.

5. If $k+1$ points are independent when taken in a certain order, they are independent when taken in any order.

6. A $(k-1)$ -space lying in a k -space separates the points of the k -space which do not lie in the $(k-1)$ -space into two sets or parts in such a way that two points of $\left\{ \begin{array}{l} \text{the same part are not} \\ \text{opposite parts are} \end{array} \right\}$ separated by the $(k-1)$ -space.

7. In a k -space, a p -space and a q -space ($p+q \geq k$) having a common point have in common precisely an r -space, where r is a definite integer such that $r \geq p+q-k$ and (by 3) $0 \leq r \leq p, 0 \leq r \leq q$.

This theorem is true for the case where $k = 1$, the 0-space being a point, as may be seen from the application of the theorem of order relations of four distinct collinear points A, B, C, D . The truth for the general k is proved by induction from k to $k + 1$.

In section 5 of his paper, Moore considers the "Hilbert" plane. He restates Hilbert's axioms $I, 7_2, 7_3$; $I, 3$; $I, 5$, and $II, 5$ in modified forms which he claims contain nothing more than Hilbert put into these axioms. The modified axioms are designated by primes. They are stated as follows:-

$I 7'_2, 7'_3$. There exists a set of three non-collinear points. Every Hilbert plane π has a notation ABC with respect to certain three non-collinear points A, B, C . In the notation ABC there is no question of order of three points A, B, C .

$I 3'$. Three non-collinear points A, B, C determine a set of points: a Hilbert plane ABC , to which they belong.

$I 5'$. With respect to a Hilbert plane ABC every line joining any

1. The first part of the report is devoted to a general survey of the work done during the year. It is divided into two main sections, the first of which deals with the work done in the laboratory and the second with the work done in the field.

2. The second part of the report is devoted to a detailed account of the work done in the laboratory. It is divided into three main sections, the first of which deals with the work done in the laboratory during the year, the second with the work done in the laboratory during the year, and the third with the work done in the laboratory during the year.

3. The third part of the report is devoted to a detailed account of the work done in the field. It is divided into three main sections, the first of which deals with the work done in the field during the year, the second with the work done in the field during the year, and the third with the work done in the field during the year.

4. The fourth part of the report is devoted to a detailed account of the work done in the laboratory during the year. It is divided into three main sections, the first of which deals with the work done in the laboratory during the year, the second with the work done in the laboratory during the year, and the third with the work done in the laboratory during the year.

5. The fifth part of the report is devoted to a detailed account of the work done in the field during the year. It is divided into three main sections, the first of which deals with the work done in the field during the year, the second with the work done in the field during the year, and the third with the work done in the field during the year.

6. The sixth part of the report is devoted to a detailed account of the work done in the laboratory during the year. It is divided into three main sections, the first of which deals with the work done in the laboratory during the year, the second with the work done in the laboratory during the year, and the third with the work done in the laboratory during the year.

7. The seventh part of the report is devoted to a detailed account of the work done in the field during the year. It is divided into three main sections, the first of which deals with the work done in the field during the year, the second with the work done in the field during the year, and the third with the work done in the field during the year.

8. The eighth part of the report is devoted to a detailed account of the work done in the laboratory during the year. It is divided into three main sections, the first of which deals with the work done in the laboratory during the year, the second with the work done in the laboratory during the year, and the third with the work done in the laboratory during the year.

9. The ninth part of the report is devoted to a detailed account of the work done in the field during the year. It is divided into three main sections, the first of which deals with the work done in the field during the year, the second with the work done in the field during the year, and the third with the work done in the field during the year.

10. The tenth part of the report is devoted to a detailed account of the work done in the laboratory during the year. It is divided into three main sections, the first of which deals with the work done in the laboratory during the year, the second with the work done in the laboratory during the year, and the third with the work done in the laboratory during the year.

point A' of the segment BC to a point D distinct from A' and lying in the plane lies wholly in the plane.

II 5'. (The plane order axiom). With respect to a Hilbert plane ABC every line which lies in the plane and which contains a point of the segment BC contains also A or B or C or a point of the segment CA or a point of the segment AB.

With the assumption that these modified axioms are the equivalent of the corresponding axioms of Hilbert, it follows at once that I,4 is deducible from the remaining axioms of group I and axioms II,1,2,3,5. The only question which may enter into this demonstration is that the modified forms of the axioms may contain more than Hilbert's corresponding axioms. In particular, this seems to me true for I,3'.

From the papers of Schur and Moore, we see that the question of what is truly axiomatic and what may be a pure matter of definition is by no means settled. I take it that the chief value arising from investigation of this nature is to show us the relative importance of certain geometrical principles, a question which is too often ignored.





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